

# The quantization of gravity and the vacuum energy of quantum fields

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## Abstract

We construct a unified covariant derivative that contains the sum of an affine connection and a Yang-Mills field. With it we construct a lagrangian that is invariant both under diffeomorphisms and Yang-Mills gauge transformations. We assume that metric and symmetric affine connection are independent quantities, and make the observation that the metric must be able to generate curvature, just as the connection, so there should be an extra tensor similar to Riemann's in the equations but constructed from metrics and not connections. We find the equations generated by the lagrangian and introduce the huge natural scale due to the vacuum energy of quantum fields. This scale allows for a perturbative solution of the equations of motion. We prove the system has a vacuum state that forces the metricity of the metric and results in General Relativity for low energies. The vacuum energy of the quantum fields cancels, becoming unobservable. At very high energies, the metric does not appear differentiated in the lagrangian and so it is not a quantum field, just a background classical field. The true quantum fields are the connections. The theory becomes very similar to a Yang-Mills, with affine connections taking the place of Yang-Mills vector fields. It should be renormalizable since it has a coupling constant with no units and correct propagators after fixing the gauge (diffeomorphisms). The weakness of gravity turns out to be due to the large vacuum energy of the quantum fields.

## 1 INTRODUCTION.

The force of gravity has been very hard to understand at the quantum level. The attempts at a covariant quantization of General Relativity (GR) present the difficulty that the coupling constant is  $\kappa = \sqrt{32\pi G} \sim 1/E_P$ , where  $E_P$  is Planck's energy and  $G$  is the gravitational constant. A coupling constant with units leads to an

unrenormalizable quantum field theory since it requires an infinite number of different counterterms for the quantum loops. This is also true in supergravity.[1] This limits somewhat the theory's usefulness although it is still possible to obtain interesting[2] results seeing it as an effective field theory.[3] At any rate this situation is a clear reminder of how limited our understanding of gravitation is.

In GR the gravitational term in the lagrangian density is proportional to the scalar of curvature, that is,  $\mathcal{L}_G \propto G^{-1}R$ . Thus the gravitational term has the units  $E^4$ , which are the correct ones for four dimensions. In the case of Yang-Mills Theories the kinetic energy in the lagrangian has the form  $F^2$ , where  $F$  is the two-form curvature associated with the Yang-Mills connection  $A$ . As a matter of fact  $F_{\mu\nu} = [D_\mu, D_\nu]$ , where  $D_\mu = \partial_\mu + A_\mu$ . The units of  $F^2$  are  $E^4$ , as in the case of the gravitational term, but now the coupling constant (hidden in the definition of  $A$  and given below) has not units. Thus we see that the cause of our troubles with gravitation is that the curvature appears to the first power in the lagrangian, and we must substitute the other curvature factor with a constant with units. It would be ideal if the gravitational term would emulate Yang-Mills Theories and have a form quadratic in the curvature. There have been many efforts to construct a gauge theory of gravity, often gauging the Poincaré group. The most common cause these efforts have not been satisfying are the difficulties involved in obtaining GR, that has to appear as a limit of some kind if we have any expectations for our theory to be realistic.

The guiding idea of this paper is that, although we really know very little about the nature of either GR or Yang-Mills Theories, we do have a clear idea of how to algebraically manipulate these two theories and what they are about: gauge invariance. Our aim is to take an intermediate step where we still maintain covariance (diffeomorphic invariance) and Yang-Mills invariance (local Lie group invariance) by means of an unified covariant derivative. This derivative is simply the sum of the Yang-Mills and the affine connections and by itself contains no new physics. But with it we can construct a curvature, and use the square of this curvature as lagrangian. The coupling constant of a theory of this kind has no units, and quantization of the theory should present no problems. But how to obtain now GR?

We assume the independence of a *symmetric* (following present understanding[4]) connection and the metric, in order to obtain a  $A_n$  manifold. Thus there is zero torsion and no metricity of the metric. The metricity of the metric, or *metricity*, for short, is the condition

$$g_{\mu\nu;\lambda} = 0. \quad (1)$$

We use this independence when we obtain the equations of motion from the lagrangian, but also in a more direct and fundamental way. Thus we make the observation that if the connection and the metric are truly independent objects, then each one should be able, in principle, to produce its own distinct contribution to the curvature of spacetime. The curvature due to the affine connection can be expressed by the familiar Riemann tensor:

$$R^\rho_{\sigma\mu\nu}[\Gamma] = \Gamma_{\sigma\mu}{}^\rho{}_{,\nu} - \Gamma_{\sigma\nu}{}^\rho{}_{,\mu} + \Gamma_{\sigma\mu}{}^\tau\Gamma_{\nu\tau}{}^\rho - \Gamma_{\sigma\nu}{}^\tau\Gamma_{\mu\tau}{}^\rho, \quad (2)$$

which is a functional of the connection (but not of the metric). The curvature due to the metric is expressed by what we shall call the metric curvature tensor (MCT):

$$\bar{R}^\rho{}_{\sigma\mu\nu}[g] = K(\delta_\nu{}^\rho g_{\sigma\mu} - \delta_\mu{}^\rho g_{\sigma\nu}), \quad (3)$$

which is a functional of the metric (but not of the connection). We then postulate that the total curvature  $\hat{R}^\rho{}_{\sigma\mu\nu}$  of the manifold is given by

$$\hat{R}^\rho{}_{\sigma\mu\nu} = R^\rho{}_{\sigma\mu\nu} + \bar{R}^\rho{}_{\sigma\mu\nu}. \quad (4)$$

The introduction of this second curvature tensor has very interesting and unexpected consequences, that arise from its special algebraic properties.

Our observational spacetime is a degenerate type of manifold compared to the one we are postulating, since in it the metric and the connection are dependent. But it is still possible to use either the connection or the metric to measure curvature in it. We can use a connection to study how a vector rotates when it is displaced parallel to itself around a closed path lying on the surface. Alternatively, we can also measure the surface's curvature using the metric, as follows: we take concentric circles and measure their radius-to-circumference ratio. If for a circle  $r/C = 1/2\pi$ , then the surface is flat; if  $2\pi r/C > 1/2\pi$ , then it is hyperbolic, and if  $2\pi r/C < 1/2\pi$ , elliptical.

The existence of a curvature due to the metric is really the only hypothesis of the paper with new content. The other one we shall require, that quantum fields have a very large vacuum energy density  $\rho_0$ , is basically accepted, the peculiar thing being that this density is not observed. If it were, the universe would be the size of a pea. We obtain the equations of motion generated by the lagrangian, and some interesting things happen: the large scale given by the vacuum energy of the quantum fields allows a perturbative solution of the equations of motion. This solution has a vacuum that makes the theory resemble GR. The small value of  $G$ , the gravitational constant, is due directly to the large value of the energy density scale, as it turns out that  $G \sim \rho_0^{-1/2}$ . Also, very conveniently, the vacuum energy density  $\rho_0$  cancels out whatever its original may be.

As long as the energy is small, that is, as long as  $E \lesssim E_P$  holds, the theory resembles GR. However, if the energies involved grow larger so that  $E \gg E_P$  holds, it is possible to neglect the MCT term and we are back to a Yang-Mills-like theory, with a coupling constant with no units and a promising Feynman diagram structure. The theory should probably be renormalizable.

In Section 2 we establish the mathematical conventions, definitions and identities we shall need; in Section 3 we define a unified covariant derivative that has both Yang-Mills and affine connections, and with it construct the lagrangian, adding, too, the MCT term; in Section 4 we solve the equations of motion and find the vacuum state of the theory; in Section 5 we study the renormalization of gravity for very high energies; in Section 6 we present our final comments. At the end there is an Appendix, where the derivation of the equations of motion from the lagrangian is done in detail.

## 2 Mathematical conventions, definitions and identities.

The conventions, definitions and identities are related to GR, Yang-Mills Theories and the MCT.

### 2.1 The conventions, definitions and identities related to General Relativity (GR).

Most of the calculations will be done on an  $n$ -dimensional spacetime. The spacetime indices are to be represented by Greek letters later in the alphabet:  $\lambda, \mu, \nu, \xi, \dots$ , with the metric  $g_{\mu\nu}$ ,  $\mu, \nu = 0, 1, 2, \dots, n-1$  having the signature  $(-+++ \dots)$ . We use  $e \equiv \sqrt{-\det(g_{\mu\nu})}$  and define the Riemann tensor as in (2):

$$\begin{aligned} R^\rho{}_{\sigma\mu\nu} &= \Gamma_{\sigma\mu}{}^\rho{}_{,\nu} - \Gamma_{\sigma\nu}{}^\rho{}_{,\mu} + \Gamma_{\sigma\mu}{}^\tau \Gamma_{\nu\tau}{}^\rho - \Gamma_{\sigma\nu}{}^\tau \Gamma_{\mu\tau}{}^\rho \\ &= \Gamma_{\sigma[\mu}{}^\rho{}_{,\nu]} + \Gamma_{\sigma[\mu}{}^\tau \Gamma_{\nu]\tau}{}^\rho. \end{aligned}$$

The Ricci tensor is  $R_{\sigma\nu} = R^\rho{}_{\sigma\rho\nu}$  and its contraction is the curvature scalar  $R$ . As usual we take the stress-energy tensor to be

$$T_{\mu\nu} = -\frac{2}{e} \frac{\delta I_M}{\delta g^{\mu\nu}}, \quad (5)$$

where  $I_M$  is the matter lagrangian. We note that  $\delta e = -\frac{1}{2} e g_{\mu\nu} \delta g^{\mu\nu}$ .

The indices of the coordinates of a tangential flat space at a point of the base spacetime manifold will be taken from the first letters of the Greek alphabet:  $\alpha, \beta, \gamma, \delta, \dots$ , the so-called non-holonomic coordinates. The tetrads will thus be written  $e^\alpha = e^\alpha{}_\mu dx^\mu$  and defined by  $g_{\mu\nu} = e^\alpha{}_\mu e^\beta{}_\nu \eta_{\alpha\beta}$ , where  $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1, \dots)$  is the Minkowski metric. Thus  $e = \det^{1/2}(-e^\alpha{}_\mu e^\beta{}_\nu \eta_{\alpha\beta}) = \det e^\alpha{}_\mu$ . The symbol  $e_\alpha{}^\mu$  is defined by  $e_\alpha{}^\mu \equiv g^{\mu\nu} \eta_{\alpha\beta} e^\beta{}_\nu$ , and has the property that  $e^\alpha{}_\mu e_\alpha{}^\nu = \delta_\mu{}^\nu$ , that is, its the inverse of the tetrad.

The Dirac matrices are defined by the algebraic condition  $\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta}$  and are coordinate-independent. The matrices defined by  $\gamma^\lambda \equiv e_\alpha{}^\lambda \gamma^\alpha$  satisfy  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$ . In terms of the tetrad, (5) can be rewritten as:[5]

$$T_{\mu\nu} = -\frac{e_{\alpha\mu}}{e} \frac{\delta I_M}{\delta e_\alpha{}^\nu}. \quad (6)$$

Vectors  $v_\beta$  undergo Lorentz rotations  $L_\alpha{}^\beta \approx \delta_\alpha{}^\beta + \omega_\alpha{}^\beta + \dots$  in the tangential space, where the  $\omega_\alpha{}^\beta$  are the antisymmetric rotation parameters, thus:  $v_\alpha \rightarrow L_\alpha{}^\beta v_\beta$ . Spinors  $\psi$  undergo spinor rotations  $S = \exp(\frac{1}{4} \omega_\alpha{}^\beta \sigma^\alpha{}_\beta)$ , where  $\sigma^\alpha{}_\beta \equiv \frac{1}{2}(\gamma^\alpha \gamma_\beta - \gamma_\beta \gamma^\alpha)$ ; thus  $\psi \rightarrow S\psi$ . As an example, if we take  $\alpha, \beta = 1, 2$ , then  $S = \exp(\frac{1}{2} \omega_1{}^2 \sigma^1{}_2)$  is the rotation due to the parameter  $\omega_1{}^2$  about the  $z$ -axis.

Since these rotations depend on the coordinates  $x^\mu$  of the point of tangency, in order to maintain invariance of the lagragian under them we introduce a spin connection defined in terms of the affine connection,

$$\omega_\alpha^\beta{}_{|\mu} \equiv e_\alpha^\lambda e^\beta_{\lambda;\mu} = e_\alpha^\lambda e^\beta_{\lambda,\mu} - e_\alpha^\lambda \Gamma_{\lambda\mu}^\nu e^\beta_\nu \quad (7)$$

where the vertical bar emphasizes that the last subindex, unlike the next two, is a holonomic coordinate. With the help of the spin connection, we can write properly invariant derivatives of vectors with non-holonomic indices and spinors. Thus for a vector field  $B_\alpha(x)$  in the tangent space to the manifold the covariant derivative is

$$D_\mu^V B_\alpha = \partial_\mu B_\alpha + \omega_\alpha^\beta{}_{|\mu} B_\beta.$$

And for a spinor field  $\psi(x)$  transforming in the spin representation of the Lorentz transformation we have the covariant derivative ( $F$  for fermion)

$$D_\mu^F \psi \rightarrow \partial_\mu \psi + \Gamma_\mu \psi, \quad \Gamma_\mu = \frac{1}{4} \sigma^\alpha_\beta \omega_\alpha^\beta{}_{|\mu}. \quad (8)$$

## 2.2 The conventions, definitions and identities related to Yang-Mills Theories.

The unitary gauge transformations of the Yang-Mills Theory will be given by  $U(x) = e^\Theta$ , where  $\Theta = -i\Theta^a T^a$ , the  $T^a$  are the group's generators and the  $\Theta^a(x)$  are the group parameters. Roman letters  $a, b, c, \dots$  will be used for the Yang-Mills indices. It will often be the case that a Dirac field transforms as

$$\psi \rightarrow U\psi$$

under a transformation  $U$ . In order to have kinetic energy terms in the lagrangian that transform properly we need a Yang-Mills covariant derivative

$$D_\mu^{YM} = \mathbf{1}_N \partial_\mu + A_\mu \quad (9)$$

where  $A_\mu = -igA_\mu^a(x)T^a$  and the  $\mathbf{1}_N$  is an  $N \times N$  identity matrix,  $N = \text{Tr } \mathbf{1}_N$  being the dimensionality of the fundamental representation of the gauge Lie group of the Yang-Mills Theory. Usually this matrix is implicitly understood, but in this paper we shall explicitly write for purposes of clarity. The  $g$  is a coupling constant with no units. We require the gauge field to transform as given by

$$A_\mu \rightarrow U A_\mu U^{-1} - (\partial_\mu U) U^{-1};$$

this way the covariant derivative will transform as:

$$D_\mu^{YM} \rightarrow U D_\mu^{YM} U^{-1}. \quad (10)$$

In this expression the differential operator is acting on all fields that may be placed to its right, and not only on  $U^{-1}$ . The field strength tensor can be obtained from the covariant derivative:

$$\begin{aligned} G_{\mu\nu} &= [D_\mu^{YM}, D_\nu^{YM}] = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) + [A_\mu, A_\nu] \\ &= A_{[\nu, \mu]} + A_{[\mu} A_{\nu]}. \end{aligned} \quad (11)$$

To work with the expression  $[D_\mu^{YM}, D_\nu^{YM}]$  one must assume that they are acting on some differentiable field placed to their right, say  $[D_\mu^{YM}, D_\nu^{YM}]\phi(x)$ . *In what follows in this paper it shall always be assumed that such differentiable fields will always be placed to the right of all covariant derivatives before any algebraic manipulation is carried out.*

### 2.3 The conventions, definitions and identities dealing with the metric curvature tensor (MCT).

The Riemann tensor carries in itself the information pertinent to the curvature of a Riemannian manifold and is constructed using connections. It has very specific symmetries between its indices related to its function as a measure of curvature. If we are to construct a tensor that gives the curvature due to the metric, there is only one way of writing it, and it is as in (3). We purport in this paper that in manifolds that have independent metric and connection the total curvature of the manifold is the sum of the curvatures due to the connection and the metric, as in (4).

The algebraic form of the MCT (3) is very familiar to us. It is the same form as that of the Riemann tensor for pseudo-Riemannian manifolds with maximal symmetry. In such manifolds it is possible to derive this form from general considerations regarding the Killing vectors and the maximal symmetry. For a signature  $(-+++ \dots)$  in  $n$  dimensions maximally symmetric manifolds are of basically only two types, up to diffeomorphisms, either de Sitter ( $K > 0$ ) or anti-de Sitter ( $K < 0$ ).

Familiar maximally symmetric ( $V_n$  Riemannian) manifolds in have a Riemann tensor defined in terms of the connection, which can be written in terms of the metric. The metricity implies that a symmetric connection is Levi-Civita. But in this paper the tensor given by (3) is a different one, defined in terms of metrics, not connections. It has little relation to the Riemann tensor of a maximally symmetric manifold. It goes without saying that the presence of the curvature term (3) as part of the curvature of a  $A_n$  manifold does not imply in any way that this manifold has to be maximally symmetric or any kind of de Sitter space.

The following are convenient definitions involving squares of the Riemann tensor and the MCT:

$$\begin{aligned} \hat{S} &\equiv \hat{R}^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} \hat{R}^\sigma_{\rho\tau\nu}, & \hat{S}_{\mu\tau} &\equiv \hat{R}^\rho_{\sigma\mu\nu} g^{\nu\nu} \hat{R}^\sigma_{\rho\tau\nu}, \\ S &\equiv R^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} R^\sigma_{\rho\tau\nu}, & S_{\mu\tau} &\equiv R^\rho_{\sigma\mu\nu} g^{\nu\nu} R^\sigma_{\rho\tau\nu}. \end{aligned} \quad (12)$$

Certain contractions involving the MCT and the Riemann tensor are going to be useful to us, so we list them here:

$$\begin{aligned}
\bar{R}_{\mu\nu} &= \bar{R}^\lambda_{\mu\lambda\nu} = -(n-1)K g_{\mu\nu}, \\
\bar{R} &= g^{\mu\nu} \bar{R}_{\mu\nu} = -n(n-1)K, \\
\bar{R}^\rho_{\sigma\mu\nu} g^{\mu\tau} \bar{R}^\sigma_{\rho\tau\nu} &= -2(n-1)K^2 g_{\nu\nu}, \\
\bar{R}^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} \bar{R}^\sigma_{\rho\tau\nu} &= -2n(n-1)K^2, \\
R^\rho_{\sigma\mu\nu} g^{\mu\tau} \bar{R}^\sigma_{\rho\tau\nu} &= 2K R_{\nu\nu}, \\
R^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} \bar{R}^\sigma_{\rho\tau\nu} &= 2K R.
\end{aligned} \tag{13}$$

Notice that some of these contractions are remarkable, especially the mixed ones of the Riemann tensor with the MCT.

### 3 A covariant derivative with both affine and Yang-Mills connections and its lagrangian.

In this Section we will construct a covariant derivative that allows us to write derivatives in lagrangians that remain invariant under both diffeomorphisms and Yang-Mills gauge transformations. We do this by the simple expedient of defining a unified covariant derivative that is the sum of the Yang-Mills and affine connections. Care should be taken with the fact that there is no metricity, so that, even if it is still possible to use the metric to lower or lift an index, it is *false* that  $B^\nu_{;\lambda} = (B_\mu g^{\mu\nu})_{;\lambda} \stackrel{?}{=} B_{\mu;\lambda} g^{\mu\nu}$ .

We begin by considering an affine connection  $\Gamma_{\mu\nu}^\lambda$  and with it defining the covariant derivative of a covariant vector

$$B_{\nu;\mu} = \partial_\mu B_\nu - \Gamma_{\mu\nu}^\lambda B_\lambda.$$

It is possible to reexpress this covariant derivative in a different way using an operator  $\tilde{D}_\mu$ , which we define by means of its components:

$$(\tilde{D}_\mu)_\nu^\lambda \equiv \partial_\mu \delta_\nu^\lambda - \Gamma_{\mu\nu}^\lambda. \tag{14}$$

The interpretation of this formula is that the connection  $\Gamma_\mu$  is a matrix with indices  $\nu$  and  $\lambda$ . When this operator  $\tilde{D}_\mu$  acts on a vector it results in the covariant derivative:

$$(\tilde{D}_\mu)_\nu^\lambda B_\lambda = \partial_\mu \delta_\nu^\lambda B_\lambda - \Gamma_{\mu\nu}^\lambda B_\lambda = B_{\nu;\mu}.$$

The similarity of (9) and (14) suggests a unified covariant derivative that has both an affine and a Yang-Mills connection:

$$(\mathfrak{D}_\mu)_\nu^\lambda \equiv \mathbf{1}_N \partial_\mu \delta_\nu^\lambda + \delta_\nu^\lambda A_\mu - \mathbf{1}_N \Gamma_{\mu\nu}^\lambda. \tag{15}$$

The symbol  $\mathbf{1}_N$  was defined in the Conventions Section. The unified covariant derivative  $\mathfrak{D}_\mu$  is a matrix that has entries both in the spacetime coordinates and in the

internal space of a representation of the compact Lie group of the Yang-Mills Theory. If it acts on a vector  $B_\lambda$  which is an element of the Lie algebra we get

$$\begin{aligned} (\mathfrak{D}_\mu)_\sigma^\lambda B_\lambda &\equiv \mathbf{1}_N \partial_\mu \delta_\sigma^\lambda B_\lambda + \delta_\sigma^\lambda A_\mu B_\lambda - \mathbf{1}_N \Gamma_{\mu\sigma}^\lambda B_\lambda \\ &= \partial_\mu B_\sigma + A_\mu B_\sigma - \Gamma_{\mu\sigma}^\lambda B_\lambda. \end{aligned} \quad (16)$$

The symbols  $\Gamma_{\mu\sigma}^\lambda$  and  $\mathbf{1}_N$  commute since the affine connection is not in the Lie algebra. On the other hand the  $A_\mu$  and the  $B_\sigma$  do not commute, being both elements of that algebra.

This unified derivative transforms as

$$\mathfrak{D}_\mu \rightarrow U \mathfrak{D}_\mu U^{-1} \quad (17)$$

under gauge Yang-Mills transformations. This follows immediately from (10) and the fact that the last term in  $\mathfrak{D}_\mu$  commutes with the Yang-Mills transformation operators  $U$ . This way  $\mathfrak{D}_\mu$  transforms suggests postulating the following lagrangian density:

$$\mathcal{L} = -\frac{1}{2ng^2} g^{\mu\tau} g^{\nu\nu} \text{Tr} \, \overline{\text{Tr}} \mathfrak{D}_{[\mu} \cdot \mathfrak{D}_{\nu]} \cdot \mathfrak{D}_{[\tau} \cdot \mathfrak{D}_{\nu]}, \quad (18)$$

where besides the usual coefficient of Yang-Mills Theories  $1/2g^2$  we have included the reciprocal of the spacetime dimension  $n$ , for reasons soon to be clear. The  $\text{Tr}$  is the usual Yang-Mills trace over the Lie algebra and the  $\overline{\text{Tr}}$  is a trace over the coordinate indices of (15). (In  $(\mathfrak{D}_\mu)_\sigma^\lambda$  the  $\sigma$  and  $\lambda$  indicate the matrix indices.) This lagrangian is invariant under diffeomorphisms because it is, by construction, a scalar. It is also invariant under the Yang-Mills gauge transformation (17) since all the unitary operators  $U$  and  $U^{-1}$  cancel among themselves.

Let us first evaluate the field strength curvature  $\mathfrak{D}^{[\mu} \cdot \mathfrak{D}^{\nu]}$ . To do the calculation we must assume that the partials are acting on some arbitrary function  $\phi$  to their right, so that we can replace  $\partial_\mu A_\nu \phi - A_\nu \partial_\mu \phi = (\partial_\mu A_\nu) \phi$ . This way we obtain:

$$\begin{aligned} [(\mathfrak{D}_\mu)_\sigma^\nu, (\mathfrak{D}_\nu)_\nu^\rho] &= (\partial_\mu \delta_\sigma^\nu + \delta_\sigma^\nu A_\mu - \Gamma_{\mu\sigma}^\nu)(\partial_\nu \delta_\nu^\rho + \delta_\nu^\rho A_\nu - \Gamma_{\nu\nu}^\rho) - \mu \leftrightarrow \nu \quad (19) \\ &= \partial_{[\mu} A_{\nu]} \delta_\sigma^\rho + [A_\mu, A_\nu] \delta_\sigma^\rho + \Gamma_{[\mu\sigma, \nu]}^\rho + \Gamma_{[\mu\sigma}^\nu \Gamma_{\nu]}^\rho \\ &= F_{\mu\nu} \delta_\sigma^\rho + \mathbf{1}_N R_{\sigma\mu\nu}^\rho. \end{aligned}$$

That is, the commutator gives a sum of the Yang-Mills field strength tensor and the Riemann tensor. To this we must add the curvature due to the metric (3). This term cannot come from any commutator of covariant derivatives since it does not depend on any connection, just on the metric. We must then add it and obtain the curvature obtained from connections:

$$F_{\mu\nu} \delta_\sigma^\rho + \mathbf{1}_N R_{\sigma\mu\nu}^\rho + \mathbf{1}_N \bar{R}_{\sigma\mu\nu}^\rho = F_{\mu\nu} \delta_\sigma^\rho + \mathbf{1}_N \hat{R}_{\sigma\mu\nu}^\rho.$$

Here we have used definition (4).



Let us go on and calculate in a more explicit form the lagrangian (18) using the results just obtained. The  $\overline{\text{Tr}}$  is not included because the initial and final indices of the right side of the equation have been summed over:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2ng^2} \text{Tr} \left( (F_{\mu\nu} \delta_\sigma^\rho + \mathbf{1}_N \hat{R}^\rho_{\sigma\mu\nu}) g^{\mu\tau} g^{\nu\nu} (F_{\tau\nu} \delta_\rho^\sigma + \mathbf{1}_N \hat{R}^\sigma_{\rho\tau\nu}) \right) \\ &= \frac{1}{2g^2} \text{Tr}(F_{\mu\nu}^2) + \frac{N}{2ng^2} \hat{R}^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} \hat{R}^\sigma_{\rho\tau\nu} \equiv \mathcal{L}_B + \mathcal{L}_G\end{aligned}\quad (20)$$

Here  $n$  is the dimensionality of spacetime and  $N$  of the fundamental representation of the Lie group. The mixed terms are zero since  $R^\sigma_{\sigma\mu\nu} = 0$ . The first term in the result,  $\mathcal{L}_B$ , is simply the Yang-Mills gauge field lagrangian. The second term,  $\mathcal{L}_G$ , has to do with gravitation.

Let us get a quick approximate estimate of the coefficient  $N/2ng^2$  using frequently used values of the quantities involved, just to get an idea of the size. Take as a grand unification group  $SU(5)$ , so  $N = 5$ , and four dimensions for spacetime, so  $n = 4$ . Assuming an asymptotic coupling constant  $\alpha(Q^2 = \infty) = 1/40$  and the usual relation  $\alpha = g^2/4\pi$ , we get  $g^2 = 4\pi/40 \approx 3$ , so  $N/2ng^2 \approx 1/5$ . This coefficient has no units and is roughly 1.

Let us assume a fermionic sector  $\mathcal{L}_F$  in the theory of the form

$$\mathcal{L}_F = \bar{\psi} \gamma^\alpha e_\alpha{}^\mu (\mathbf{1}_N \partial_\mu + \mathbf{1}_N \Gamma_\mu + A_\mu) \psi \quad (21)$$

where the fermion fields  $\psi$  are chiral and transform in a complex representation of the Lie group, and where the symbol  $\Gamma_\mu$  is defined in (8). The correct fermionic lagrangian is a symmetrization of the one above. I will not write the symmetrizations explicitly in this paper to keep related equations and derivations simpler. The correct *symmetrized* result for the fermionic stress-energy tensor is explicitly written in [5]. The total action of this theory is then the sum of three terms: gravitational, bosonic and fermionic

$$I_T = \int e(\mathcal{L}_B + \mathcal{L}_G + \mathcal{L}_F) d^n x \quad (22)$$

Much of the point of this paper has to do with the solution to the equations of motion generated by the first-order variations of the action  $I$  with respect to the metric  $g_{\mu\nu}$  and the connection  $\Gamma_{\mu\nu}{}^\lambda$ . There is a substantial amount of algebra involved in finding these equations of motion. The algebraic development is given in the Appendix. In next Section we take the equations from the Appendix and proceed directly to solve them.

## 4 First order solution of the two equations of motion: emergence of General Relativity (GR).

A perennial theoretical problem in quantum field theory is its clear prediction of a huge vacuum energy due to quantum fluctuations. We know that the virtual processes

predicted by quantum field theory do exist, since their contributions to experimentally observed quantities in the standard model of high energy physics are necessary in order to have consistency with experiment. On the other hand, for such a huge energy density, Einstein field equation predicts a tiny universe. So we are in the peculiar situation of having a clear-cut prediction from quantum field theory, the most successful theory in physics, not being observed. Our aim here is to kill two birds with one stone, by using this huge energy density as a large scale to be able to perturbatively solve the equations of motion, while at the same time giving an explanation for its inconspicuousness.

The vacuum energy acts as a constant energy density  $\rho_0$ , so that the stress-energy tensor can be written in the form  $T_{\rho\tau} = \rho_0 g_{\rho\tau} + T'_{\rho\tau}$ , where  $T'_{\rho\tau}$  is the stress-energy tensor with the vacuum energy subtracted. The stress-energy tensor should satisfy  $\langle 0|T_{\rho\tau}|0\rangle = \rho_0 g_{\rho\tau}$ . In the Appendix we have derived the two equations of motion generated by action (22) for the independent first variations of the metric and the connection. Collecting these equations from the Appendix:

$$\frac{N}{ng^2} \left( 2S_{\rho\tau} - \frac{1}{2}Sg_{\rho\tau} + 4K(R_{\rho\tau} - \frac{1}{2}Rg_{\rho\tau}) + K^2n(n-1)g_{\rho\tau} \right) = \rho_0 g_{\rho\tau} + T'_{\rho\tau}, \quad (23)$$

$$\text{and} \quad \frac{8N}{ng^2} (eR^\rho_{\sigma\mu\nu}g^{\mu\tau}g^{\nu\nu} + e\bar{R}^\rho_{\sigma\mu\nu}g^{\mu\tau}g^{\nu\nu})_{;v} + e\bar{\psi}\gamma^\tau\sigma^\rho_\sigma\psi = 0, \quad (24)$$

where we have expanded the stress-energy tensor into vacuum and real matter parts. Notice the appearance of Einstein's equation left side in the first equation.

#### 4.1 Solution of the first equation of motion (23).

It is accepted lore nowadays that, theoretically, the vacuum density should be so large that

$$\langle T'_{00} \rangle / |\rho_0| \sim 10^{-123}.$$

This estimate comes from assuming that  $\langle T'_{00} \rangle$  is of the order of the critical density of the universe, and that the vacuum energy density of the quantum fields would be given by some quantum theory that uses Plank's constant, so that  $\rho_0 \sim G^{-2}$ . Let us assume only that  $\rho_0 \gg |T'_{00}|$  (but not that  $\rho_0 \sim G^{-2}$ ) and use this relation to establish a large scale to solve the equations of motion perturbatively.

In Eq. (23) there are two terms explicitly involving the metric, and we assume that they cancel each, so that, up to order  $K^2$ ,

$$(n-1)\frac{N}{g^2}K^2 = \rho_0. \quad (25)$$

(Remember that  $K$  is the constant in the MCT (3).) Since  $N \sim n \sim g \sim 1$ , we conclude that  $K^2 \sim \rho_0$ , so that  $K$  is huge, too, with respect to the other curvature terms and  $T'_{\rho\tau}$ . Having established  $K$  as a large scale, we get, to order  $K$ , Einstein's equation

$$R_{\rho\tau} - \frac{1}{2}Rg_{\rho\tau} = -8\pi GT'_{\rho\tau},$$

assuming we have made the identification

$$-8\pi G = \frac{ng^2}{4KN}. \quad (26)$$

(Here we are neglecting the terms  $2S_{\rho\tau} - \frac{1}{2}Sg_{\rho\tau}$  because they consist of weak gravitational fields squared, and they are not multiplied by the large factor  $K$ , like the other terms of the left-hand side of (23).) This identification makes sense since it implies  $K^2 \sim G^{-2}$ , thus verifying that  $K$  is very large. Furthermore, these two relations allow us to make the prediction

$$\rho_0 = \frac{(n-1)n^2g^2}{1024\pi^2N} \cdot \frac{1}{G^2},$$

or, order-of-magnitude,  $\rho_0 \sim G^{-2}$ . This relation is a prediction of the model. We have obtained it using only the hypothesis that  $\rho_0 \gg |T'_{00}|$ , no more.

## 4.2 Solution to the second equation of motion (24).

Let us pay attention now to the second equation of motion (24). Let us write this equation in the form

$$K (eg_{\sigma[\mu}\delta_{\nu]}^{\rho}g^{\mu\tau}g^{\nu\nu})_{;v} + (eR^{\rho}_{\sigma\mu\nu}g^{\mu\tau}g^{\nu\nu})_{;v} + \frac{ng^2}{8N}e\bar{\psi}\gamma^{\tau}\sigma^{\rho}_{\sigma}\psi = 0, \quad (27)$$

where we have used (3). In order for the equation to be satisfied, considering that both the second and third terms are very small, the coefficient of the very large scale factor  $K$  has to be zero, or

$$(eg_{\sigma\mu}\delta_{\nu}^{\rho}g^{\mu\tau}g^{\nu\nu} - eg_{\sigma\nu}\delta_{\mu}^{\rho}g^{\mu\tau}g^{\nu\nu})_{;v} = 0. \quad (28)$$

This equation implies the metricity (1). A resulting equation is left that relates the second and third terms of Eq. (27) and the spin of elementary particles with the divergence of Riemann's tensor. For macroscopic purposes the spin term should be an expectation value,  $\langle\bar{\psi}\gamma^{\tau}\sigma^{\rho}_{\sigma}\psi\rangle$ . The quantum vacuum expectation value of this term is zero, since the spin contributions from all the particles should cancel. As far as matter goes, the expectation value is simply a density average value. For the solar system this term is basically negligible. However, for galaxies with large quantities of interstellar gases and plasmas, especially in the presence of galactic magnetic fields capable of polarizing them, there should be a strong coupling of the polarized quantum spin with the divergence of the Riemann curvature tensor, according to this model.

Perhaps it would be clarifying here to recall the situation when the Palatini variation is taken in the case of GR. In this case (if one accepts the presence of fermion fields and that the spin connection of these fields depends on the affine connection), an equation similar to (27) is found:

$$-\frac{1}{16\pi G} \left[ \frac{1}{2}(eg^{\sigma\nu})_{;\nu}\delta^{\rho}_{\tau} + \frac{1}{2}(eg^{\rho\nu})_{;\nu}\delta^{\sigma}_{\tau} - (eg^{\sigma\rho})_{;\tau} \right] = e\bar{\psi}\gamma^{\rho}\sigma^{\sigma}_{\tau}\psi.$$

The large value of the coefficient  $1/G$  on the left and the small value of the quantity on the right assures us that the quantity in square brackets is basically zero, a situation that can only occur if  $g_{\mu\nu;\lambda} = 0$ . Thus the manifold possesses metricity again, and we have obtained a theory *classically* similar to GR. Thus the Palatini variation with respect to the connection in this model results in an equation similar to the same variation in the standard metric-affine model.

Going back to Eqs. (27) and (28), in the absence of matter one could simply set

$$R^\rho{}_{\sigma\mu\nu}{}^{;\nu} = 0, \quad (29)$$

where we have used the metricity to simplify the equation. At first sight this would seem to certainly result in a theory very different from GR. Actually, this is not the case as we shall see. We will first derive an implication of (29) that is related to Einstein's equation. Take (29) and contract  $\rho$  and  $\mu$  to get  $R_{\sigma\nu}{}^{;\nu} = 0$ . Now the Bianchi identities can be written in the form  $R_{\sigma\nu}{}^{;\nu} - \frac{1}{2}g_{\sigma\nu}R^{;\nu} = 0$ , so we conclude from them and the previous equation that  $R^{;\nu} = 0$ , that is, *the curvature scalar has to be constant*. This condition does not exist in GR in general but GR several solutions upheld it anyway.

In the table that follows we list some representative solutions of Einstein's equation, with their values for  $R^\lambda{}_{\mu\nu\kappa;\lambda}$ ,  $R^{;\nu}$ , and  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$  given for all points except perhaps a null measure set:

Solution	$R^\lambda{}_{\mu\nu\kappa;\lambda}$	$R^{;\nu}$	$G_{\mu\nu}$
Schwarzschild	0	0	0
Kerr	0	0	0
Reissner-Nordstrom	Not 0	0	Not 0
Kerr-Newman	Not 0	0	Not 0
Robertson-Walker inflation $\kappa = 0$	0	0	Not 0
Robertson-Walker dust $\kappa = 0$	Not 0	Not 0	Not 0

Thus there is immediate agreement for three of those solutions of the GR with the model presented here: Schwarzschild, Kerr, and Robertson Walker inflation with flat space  $\kappa = 0$ . In particular, there is agreement in the Schwarzschild solution case, which accounts for most of the accurate verifications of GR. There is no agreement for some of the solutions, but that is not necessarily an argument against the model, since in those particular cases we are not certain of the correction of the very large scale behavior of GR. Those differences could be useful explaining a component of dark matter, or finding a dynamical explanation for dark energy.

We thus arrive at a theory that is macroscopically similar (although not identical) to GR, and one which enforces metricity.

## 5 Quantization of the theory: its high energy limit.

We have presented a model in this paper that has as a low energy limit GR. We now study the problem of the renormalization of this theory. This is essentially an

ultraviolet problem so we will be concerned only with very high energies  $E \gg E_P$ . Yes, in this model Planck's energy is the low energy limit. Our starting point is the original lagrangian (20) of the model, from where we get for pure gravity

$$\mathcal{L}_G = \frac{N}{2ng^2} (R^\rho_{\sigma\mu\nu} + \bar{R}^\rho_{\sigma\mu\nu}) g^{\mu\tau} g^{\nu\nu} (R^\sigma_{\rho\tau\nu} + \bar{R}^\sigma_{\rho\tau\nu}). \quad (30)$$

The scale of the energy of the  $\bar{R}^\rho_{\sigma\mu\nu}$  term is  $|K|^{1/2} \sim G^{-1/2} \sim E_P$ . While this is certainly a large energy scale we can consider even higher energies  $E \gg E_P$ . In this very high energy limit the independence of the metric and the connections is restored. The point to notice here is that neither the MCT nor the Riemann tensor contain derivatives of the metric. As a matter of fact there are not any derivatives of the metric anywhere in the lagrangian. Thus for very high energies the terms involving the connection keep gaining kinetic energy, while terms involving the metric have stagnant energies. Eventually what is going to occur is that  $|R^\rho_{\sigma\mu\nu}| \gg |\bar{R}^\rho_{\sigma\mu\nu}|$ . So we can simplify Eq. (30) and just write

$$\mathcal{L}_{G \text{ very high energy}} = \frac{N}{2ng^2} R^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} R^\sigma_{\rho\tau\nu}. \quad (31)$$

In this model the metric is not a quantum field, it is always classical. It does not have a canonical conjugate field to form a quantum commutator with. The quantum field of gravity in this model is the affine connection that makes up the Riemann tensor. Its kinetic energy terms are

$$\mathcal{L}_G = \frac{N}{2ng^2} (\Gamma_{\sigma\mu}{}^\rho{}_{,\nu} - \Gamma_{\sigma\nu}{}^\rho{}_{,\mu}) g^{\mu\tau} g^{\nu\nu} (\Gamma_{\rho\tau}{}^\sigma{}_{,\nu} - \Gamma_{\rho\nu}{}^\sigma{}_{,\tau})$$

Let us find the free equation of motion of the connection. We do this by taking a first variation with respect to  $\Gamma_{\sigma\mu}{}^\rho$ :

$$\delta \int e \mathcal{L}_G d^n x = \frac{N}{ng^2} \int e (\Gamma_{\sigma\mu}{}^\rho{}_{,\nu} - \Gamma_{\sigma\nu}{}^\rho{}_{,\mu}) g^{\mu\tau} g^{\nu\nu} \delta \Gamma_{\rho\tau}{}^\sigma{}_{,\nu} d^n x$$

At these energies the metric is a classical background field changing at a far smaller rate than the quantum excitations (the connections) and we simply take it as constant. Applying parts we can then obtain the equation of motion the connections obey:

$$\Gamma_{\sigma\mu}{}^\rho{}_{,\nu}{}^{,\nu} - \Gamma_{\sigma\nu}{}^\rho{}_{,\mu}{}^{,\nu} = 0. \quad (32)$$

We can take advantage of the diffeomorphism gauge invariance and set  $\Gamma_{\sigma\nu}{}^\rho{}_{,\mu}{}^{,\nu} = 0$ , so that the equation simply becomes the wave equation  $\Gamma_{\sigma\mu}{}^\rho{}_{,\nu}{}^{,\nu} = 0$ . The connections travel like other quantum fields.

For purposes of quantization the gauge invariance has to be dealt with in a different fashion. One way of dealing with this problem is adding the gauge-fixing term

$$\mathcal{L}_{\text{fix}} = -\frac{\lambda}{2} (\partial^\mu \Gamma_{\sigma\mu}{}^\rho) (\partial^\nu \Gamma_{\rho\nu}{}^\sigma)$$

to the pure gravity lagrangian. Thus the matrix operator (in square brackets) in the resulting lagrangian

$$\mathcal{L}_G + \mathcal{L}_{\text{fix}} = \frac{1}{2} \Gamma_{\sigma\mu}{}^\rho [g^{\mu\nu} \partial^2 - (1 - \lambda) \partial^\mu \partial^\nu] \Gamma_{\rho\nu}{}^\sigma$$

is no longer singular (its singular nature is due to the diffeomorphism invariance) and can be inverted, resulting in a convenient propagator. This, plus the fact that the coupling constant has no units, gives very good perspectives for quantization.

## 6 Final comments.

We have constructed a unified covariant derivative using a Yang-Mills connection and an affine connection summed together. We then assume that the affine connection and the metric are independent, and make the observation that, just as the connection can generate curvature (measured by the Riemann tensor), so could the metric produce a curvature (measured by the MCT, the metric curvature tensor). A long but straightforward chain of mathematical steps and the large scale of the vacuum energy due to quantum fields result, in a low energy regime, in GR. For very high energies the MCT can be disregarded and we get for pure gravity a theory very similar to a Yang-Mills one, with the affine connections playing the role of vector gauge fields. These connections are the quantum fields of gravity, and the metric is just a background classical field at this energy level. The coupling constant has no units and therefore the theory is probably renormalizable. Since it mimics the Feynman diagram structure of a Yang-Mills Theory, it probably preserves unitarity, too.

A comment about unitarity. When Levi-Civita connections enter in a lagrangian a way similar to Yang-Mills fields, the metrics that make up these connections appear in complicated products of powers of first and second order derivatives of the metric,[6] greatly complicating the unitarity issue.[7] In the model presented in this paper, at very high energies all the metrics that appear are not being differentiated, and therefore the metric is just a classical background field. During the quantization process in the ultraviolet regime the uncomfortable powers of derivatives of the metric simply do not appear at all.

Actually, the theory that appears in the low energy regime is not exactly GR. As studied in Section 4 it has some solutions that are the same as GR, while others are different. In particular, the Schwarzschild solution is the same for both models. Since it is this solution that has most of the experimental backing, this model does not have conflict with macroscopic observation. As a matter of fact, its extra freedom in large structure solutions is welcome as observation on large scales has resulted in a rather unclear experimental and theoretical situations in GR, to witness, dark energy and dark matter.[8]

This model is conceptually and mathematically simple, yet raises the possibility of a quantizable gravity. It has the added merit of giving a mechanism that allows for the vanishing of the vacuum energy density of the quantum fields. It relates

in a unequivocal logical way the large value of this vacuum energy density and the weakness of  $G$ , the gravitational constant. It also gives a link between Yang-Mills Theories and gravity (remember the Yang-Mills coupling constant  $g$  is entering in gravitational equations) but the relation seems incidental and does not shed light at all on the origin of Yang-Mills Theories.

The arguments given in the previous Section on quantization explain why at very high energies some terms dominate over others. If we assume very high temperatures, like they existed during the Big Bang, it is possible to introduce a chemical potential and the formalism of finite-temperature quantum field theory. In this case the chemical potential increases with the temperature and eventually the vacuum that is responsible for the system acting as GR becomes metastable. When the system falls into the real vacuum of the theory it will resemble the ultraviolet behavior studied in last Section.

## Appendix A. The derivation of the equations of motion from the total lagrangian.

In this appendix we find the equations of motion that result from taking a first variation with respect to the affine connection and the metric of the total action of this model (22):

$$I_T = I_G + I_M \quad (33)$$

where

$$I_G = \frac{N}{2ng^2} \int e \hat{R}^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} \hat{R}^\sigma_{\rho\tau\nu} d^n x = \frac{N}{2ng^2} \int e \hat{S} d^n x$$

and

$$I_M = \int e (\mathcal{L}_B + \mathcal{L}_F) d^n x. \quad (34)$$

Recall the definition of  $\hat{R}^\rho_{\sigma\mu\nu}$  is given by (4) and of  $\hat{S}$  by (12).

1) *We consider first the variation of  $I_T$  with respect to the metric.*

We proceed by calculating the first variation of  $e\hat{S}$  with respect to the metric. The metric appears in  $e$ ; as usual  $\delta e = -\frac{1}{2}e g_{\mu\tau} \delta g^{\mu\tau}$ , so that the variation in  $e$  due to the metric is

$$\begin{aligned} \hat{S} \delta e &= -\frac{1}{2} e \hat{S} g_{\mu\tau} \delta g^{\mu\tau} \\ &= -\frac{1}{2} e (S + 4RK - 2K^2 n(n-1)) g_{\mu\tau} \delta g^{\mu\tau}, \end{aligned}$$

where we have used the identities in (13). The metric also appears in the symbol  $\hat{S}$  of Eq. (12), explicitly performing the product between the two curvature tensors, and the variation in  $\hat{S}$  due to this presence is

$$\begin{aligned} \delta \hat{S}_1 &= 2 \hat{S}_{\mu\tau} \delta g^{\mu\tau} \\ &= 2 (S_{\mu\tau} + 4K R_{\mu\tau} - 2K^2 (n-1) g_{\mu\tau}) \delta g^{\mu\tau}, \end{aligned}$$

where we have used again identities (13). The last place the metric appears is in the MCT  $\bar{R}^\rho_{\sigma\mu\nu}$  (but it never appears in the  $R^\rho_{\sigma\mu\nu}$ ). This first variation can be calculated as follows. Use  $\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$  to verify that

$$\delta \bar{R}^\rho_{\sigma\mu\nu} = K\delta_\nu{}^\rho \delta g_{\sigma\mu} - K\delta_\mu{}^\rho \delta g_{\sigma\nu} = -K\delta_{[\nu}{}^\rho g_{\mu]\lambda} g_{\sigma\nu} \delta g^{\nu\lambda}.$$

With this result prove that

$$\begin{aligned} \delta \hat{S}_2 &= 2\delta \bar{R}^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} (R^\sigma_{\rho\tau\nu} + \bar{R}^\sigma_{\rho\tau\nu}) \\ &= -4K (R_{\mu\tau} - (n-1)K g_{\mu\tau}) \delta g^{\mu\tau}. \end{aligned} \quad (35)$$

The total variation is given  $\delta(e\hat{S}) = \hat{S}\delta e + e\delta\hat{S}_1 + e\delta\hat{S}_2$ . To find the equation of motion we must consider the variation of  $I_M$  with respect to the metric. This can be done using (5) and (6), and setting

$$T_{\mu\nu} = -\frac{2}{e} \frac{\delta I_M}{\delta g^{\mu\nu}},$$

where  $I_M$  is given by (34). The sum of the gravitational and the matter variations gives:

$$\frac{N}{2ng^2} \int \delta(e\hat{S}) d^n x - \frac{1}{2} \int e T_{\mu\nu} \delta g^{\mu\nu} = 0,$$

from which we obtain the equation of motion generated by the metric:

$$\frac{N}{ng^2} \left( 2S_{\rho\tau} - \frac{1}{2} S g_{\rho\tau} + 4K(R_{\rho\tau} - \frac{1}{2} R g_{\rho\tau}) + K^2 n(n-1) g_{\rho\tau} \right) = T_{\rho\tau}. \quad (36)$$

2) *We consider now the first variation of  $I_T$  with respect to the connection.* The connection appears in  $I_T$  only in two places: in the Riemann curvature tensor, and in the spin connection that appears in  $I_M$  in the lagrangian  $\mathcal{L}_F$  shown in (21). The spin connection (7) contains the affine connection in the covariant derivative. As before, we proceed first to study the variation of  $e\hat{S}$ , in this case with respect to the connection.

Notice that

$$\delta \hat{R}^\sigma_{\rho\tau\nu} = \delta R^\sigma_{\rho\tau\nu} + \delta \bar{R}^\sigma_{\rho\tau\nu} = \delta R^\sigma_{\rho\tau\nu} = (\delta \Gamma_{\rho[\tau}{}^\sigma{}_{\nu]}).$$

The first variation of the MCT is zero,  $\delta \bar{R}^\sigma_{\rho\tau\nu} = 0$ , since it is a functional of the metric only, and the last step in the Eq. above relies on Palatini's identity. Then the variation of the gravitational term of the actions is:

$$\begin{aligned} \delta I_G &= \frac{N}{ng^2} \int e \hat{R}^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} (\delta \Gamma_{\rho[\tau}{}^\sigma{}_{\nu]}) d^n x, \\ &= -2 \frac{N}{ng^2} \int \left( e \hat{R}^\rho_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} \right)_{;\nu} \delta \Gamma_{\rho\tau}{}^\sigma d^n x, \end{aligned} \quad (37)$$

where in the last step we have used ‘‘covariant parts’’.



We now confront the variation of the matter term  $I_M$  of the action. (The lagragian density  $\mathcal{L}_B$  is assumed to consist of Yang-Mills and similar theories that do not depend on the affine connection. In Yang-Mills Theories all the covariant derivatives appear in the form  $A_{[\mu;\nu]} = A_{[\mu,\nu]}$ , that does not contain connections.) We calculate this variation through the functional derivative

$$\begin{aligned}\frac{\delta I_M}{\delta \Gamma_{\rho\tau}{}^\sigma} &= \int \frac{\delta e \mathcal{L}_F}{\delta \Gamma_{\rho\tau}{}^\sigma} d^n x = \int e \bar{\psi} \gamma^\mu \frac{\delta \Gamma_\mu}{\delta \Gamma_{\rho\tau}{}^\sigma} \psi d^n x \\ &= \frac{1}{4} \int e \bar{\psi} \gamma^\mu \sigma^\alpha{}_\beta \frac{\delta e_\alpha{}^\lambda e^\beta{}_{\lambda;\mu}}{\delta \Gamma_{\rho\tau}{}^\sigma} \psi d^n x = -\frac{1}{4} e \bar{\psi} \gamma^\tau \sigma^\rho{}_\sigma \psi,\end{aligned}$$

using the definitions (7) and (8) and the convention of letters from the beginning and the middle of the Greek alphabet set up in Section 2. With this result and our previous (37) we get the equation of motion generated by the connection:

$$\frac{8N}{ng^2} \left( e \hat{R}^\rho{}_{\sigma\mu\nu} g^{\mu\tau} g^{\nu\nu} \right)_{;v} = -e \bar{\psi} \gamma^\tau \sigma^\rho{}_\sigma \psi. \quad (38)$$

We have obtained the two coupled equations of motion that result from a first variation of the action (33) with respect to the connection and the metric. In Section 4 above we find an interesting perturbative solution to them.

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